INTRODUCTION TO RELATIVISTIC COSMOLOGY

9-1 INTRODUCTION

In the general theory of relativity, Einstein set out to identify the gravitational field with the geometry of space time itself. Using differential geometry, he derived an equation which identified the curvature of space time with the matter-energy density contained in it. Expressed in tensor form, the Einstein field equations are written

$$G_{\mu\nu}-\tfrac{1}{2}(Gg_{\mu\nu}-2\Lambda g_{\mu\nu})=8\pi\gamma T_{\mu\nu} \tag{9-1}$$

where $G_{\mu\nu}$ is the Einstein-Ricci tensor, G is its trace, $g_{\mu\nu}$ is the metric tensor which is defined by the equation

$$ds^2=g_{\mu\nu}\,dx_\mu\,dx_\nu \tag{9-2}$$

μ and ν are tensor indices which take on four values (one timelike, three spacelike), Λ is the cosmological constant, γ is the gravitational constant, and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu}=\left(\frac{p}{c^2}+\rho\right)\beta_\mu\beta_\nu+\frac{p}{c^2}\,g_{\mu\nu} \tag{9-3}$$

for a macroscopic body. In equation (9–1), p is the pressure, ρ the mass density, and β_μ and β_ν are components of the comoving velocity vector within the mass.

For an isotropic and homogeneous universe, the metric tensor has the Robertson-Walker form in dimensionless spherical coordinates; i.e.,

$$g_{\mu\nu}=\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & -R(t)(1-k\eta^2)^{-1/2} & 0 & 0 \\ 0 & 0 & -R(t)\eta & 0 \\ 0 & 0 & 0 & -R(t)\eta\sin\theta \end{pmatrix} \tag{9-4}$$

where η is a dimensionless radial coordinate. Thus

$$ds^2 = c^2\,dt^2 - R^2(t)\left[(1-k\eta^2)^{-1}\,d\eta^2 + \eta^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \tag{9–5}$$

where $R(t)$ is a time-dependent scale factor to be determined by solving the field equations (eq. (9–1)), and k is the curvature constant which has the values -1, 0, and $+1$.

Using the Robertson-Walker metric, equation (9–1) simplifies into the relations

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{8\pi\gamma p}{c^2} = -\frac{kc^2}{R^2} + \Lambda c^2 \tag{9–6}$$

and

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi\gamma\rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{9–7}$$

In the present state of the universe, $\rho \gg p/c^2$, and equations (9–6) and (9–7) are usually solved for the zero-pressure approximation.

The Robertson-Walker line element given in equation (9–5) can also be written in the form

$$ds^2 = c^2\,dt^2 - R^2(t)\,du^2 \tag{9–8}$$

where

$$du^2 = \frac{d\eta^2}{1-k\eta^2} + \eta^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{9–9}$$

The dimensionless length measured along the radial direction is then given by

$$u = \int_0^{\eta} \frac{d\eta}{(1-k\eta^2)^{1/2}} = \begin{cases} \sin^{-1}\eta & \text{for } k=+1 \\ \eta & \text{for } k=0 \\ \sinh^{-1}\eta & \text{for } k=-1 \end{cases} \tag{9–10}$$

Photons travel along geodesics which obey the same relation as in special relativity; viz,

$$ds^2 = 0 \tag{9–11}$$

which, from equation (9–8), is equivalent to the condition

$$u = c\int_{t_e}^{t_r} \frac{dt}{R(t)} \tag{9–12}$$

with t_e being the time when the photon was emitted and t_r being the time when the photon was received.

The emitting and receiving points are embedded in the metric so that the distance between them is changing by the scale factor $R(t)$; the dimensionless metric distance u is a constant. If we therefore consider

two successive wave crests of a light ray as being emitted at times t_e and $t_e+\Delta t_e$, respectively, and received at times t_r and $t_r+\Delta t_r$, then

$$\int_{t_e}^{t_r}\frac{dt}{R(t)}=\int_{t_e+\Delta t_e}^{t_r+\Delta t_r}\frac{dt}{R(t)}=u=\text{constant} \tag{9–13}$$

Thus

$$\int_{t_e+\Delta t_e}^{t_r+\Delta t_r}\frac{dt}{R(t)}-\int_{t_e}^{t_r}\frac{dt}{R(t)}=\int_{t_r}^{t_r+\Delta t_r}\frac{dt}{R(t)}-\int_{t_e}^{t_e+\Delta t_e}\frac{dt}{R(t)}$$

$$=\frac{\Delta t_r}{R(t_r)}-\frac{\Delta t_e}{R(t_e)}=0 \tag{9–14}$$

or

$$\frac{\Delta t_r}{R(t_r)}=\frac{\Delta t_e}{R(t_e)}$$

Since the wavelength of the emitted wave is $c\Delta t_e$ and that of the wave when received is $c\Delta t_r$, from equation (9–14) it follows that the wavelength is shifted by the amount

$$z\equiv\frac{\lambda_r-\lambda_e}{\lambda_e}=\frac{\Delta\lambda}{\lambda}=\frac{R(t_r)-R(t_e)}{R(t_e)}$$

or

$$\frac{R(t_r)}{R(t_e)}=1+z \tag{9–15}$$

We have observed this shift in the spectral lines of distant galaxies as always being toward longer wavelengths, so that $R(t_r)>R(t_e)$. From this evidence, it has been deduced that our universe is expanding with time.

9–2 GAMMA-RAY FLUXES

Let us now consider the effect of cosmological factors in calculating γ-ray fluxes emitted at large redshifts, z. The number of photons received per second is reduced by a factor $R(t_e)/R(t_r)$ from the number produced per second at time t_e. We consider here γ-rays produced in particle collisions between two components having densities $n_a(t_e)$ and $n_b(t_e)$, respectively. We specify the differential photon intensity produced per collision as

$$G(E_\gamma) \qquad (\text{cm}^2\text{-s-sr-GeV-cm}^{-6})^{-1}$$

Then the differential photon flux received at t_r is given by

$$dF_r=\frac{4\pi n_a(t_e)n_b(t_e)G(E_{\gamma,e})\,dE_{\gamma,e}\,dV_e\,dt_e}{4\pi R^2(t_r)\eta^2} \tag{9–16}$$

where the numerator represents the photon flux emitted at t_e, and the denominator indicates the fact that at t_r this flux is evenly distributed over a spherical wavefront of radius $R(t_r)$.

We now define the dimensional length

$$dl = R(t)\ du \tag{9–17}$$

so that the volume element

$$dV_e = dl[R^2(t_e)\eta^2\ d\Omega] \tag{9–18}$$

Since

$$dt_e = \frac{R(t_e)}{R(t_r)}\ dt_r$$

and

$$dE_{\gamma,e} = \frac{R(t_r)}{R(t_e)}\ dE_{\gamma,r} \tag{9–19}$$

because the energy of a γ-ray is inversely proportional to its wavelength, we may substitute equations (9–18) and (9–19) into equation (9–17) and obtain

$$dF_r = n_a(t_e)\, n_b(t_e) G\left[\frac{R(t_r)}{R(t_e)}\ E_{\gamma,r}\right] \times \frac{4\pi R^2(t_e)\eta^2\ d\Omega\ dl\ dE_{\gamma,r}\ dt_r}{4\pi R^2(t_r)\eta^2} \tag{9–20}$$

Dropping the subscript r from equation (9–20), since γ-rays are measured only when they are received, we may reduce equation (9–20) to

$$\frac{dF}{d\Omega\ dt\ dE_\gamma} \equiv dI = \frac{n_a(z) n_b(z) G[(1+z)E_\gamma] dl}{(1+z)^2}$$

$$= \frac{n_a(z) n_b(z) G[(1+z)E_\gamma]}{(1+z)^2}\left(\frac{dl}{dz}\right) dz \tag{9–21}$$

by using the relation (9–15).

9–3 DERIVATION OF *dl/dz*

Equation (9–21) is quite useful in evaluating the metagalactic γ-ray spectra from various high-energy interactions. The results are obtained from numerical integration of the relation

$$I(E_\gamma) = \int_0^{z_{\max}} dz\ n_a(z) n_b(z)\ \frac{G[(1+z)E_\gamma]}{(1+z)^2}\left(\frac{dl}{dz}\right) \tag{9–22}$$

Therefore, in order to utilize equation (9–22), we must determine the

factor dl/dz by solving the field equations (9–6) and (9–7) for particular cases. As we shall discuss later, the two most important cases to consider are the Einstein-de Sitter model and the low-density model. We shall now evaluate dl/dz for these models and for the more general case.

9–3a The Einstein-de Sitter Model

This model corresponds to a Euclidean three-space with $k = \Lambda = p = 0$. Equation (9–7) then reduces to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi\gamma\rho}{z} \tag{9–23}$$

In this case, our space is a Euclidean three-sphere containing a constant mass-energy M, so that

$$\rho \sim \frac{M}{R^3} \tag{9–24}$$

From equations (9–23) and (9–24) we find

$$R(t) \sim t^{2/3} \tag{9–25}$$

In these equations, it is common to define the Hubble parameter

$$H \equiv \frac{\dot{R}}{R} \tag{9–26}$$

so that from equations (9–25) and (9–26), we obtain

$$H = \tfrac{2}{3} t^{-1} \tag{9–27}$$

It is also common to label quantities associated with the present epoch ($z = 0$) by a subscript 0. The quantity H_0 is then referred to as the Hubble constant. From equations (9–15), (9–25), and (9–27), we then find

$$R(z) = \frac{R_0}{1+z}$$

and (9–28)

$$t(z) = \frac{t_0}{(1+z)^{3/2}}$$

From equations (9–8), (9–11), and (9–17), we obtain

$$\frac{dl}{dz} = -c\frac{dt}{dz} \tag{9–29}$$

and from equations (9–27), (9–28), and (9–29), we obtain

$$\frac{dl}{dz} = \frac{cH_0^{-1}}{(1+z)^{3/2}} \tag{9–30}$$

9-3b The Low-Density Model

This model holds when $\Lambda = 0$ and $\frac{8\pi\gamma\rho R^2}{3c^2} \ll 1$. Thus, equation (9–7) reduces to

$$\frac{\dot{R}^2}{R^2} = -\frac{kc^2}{R^2} \tag{9–31}$$

For a physical solution, we must specify $k = -1$ (negative curvature) and, therefore, $\dot{R} =$ constant and $R = Kt$, where $K =$ constant. Therefore

$$t(z) = \frac{t_0}{1+z}$$

$$H(z) = H_0(1+z) \tag{9–32}$$

$$H_0 = t_0^{-1}$$

and

$$\frac{dt}{dz} = -\frac{t_0}{(1+z)^2}$$

From equations (9–29) and (9–32), we then find

$$\frac{dl}{dz} = \frac{cH_0^{-1}}{(1+z)^2} \tag{9–33}$$

9-3c Other Models With $\Lambda = p = 0$

For models of a more general form with $\Lambda = p = 0$, but where $k \neq 0$ and $\rho \neq 0$, it is useful to define a new parameter

$$\Omega \equiv -\frac{2}{H_0^2}\frac{\ddot{R}_0}{R_0} \tag{9–34}$$

which provides a measure of the gravitational deceleration of the universal expansion caused by the presence of matter of density ρ. By subtracting equation (9–7) from equation (9–6), we find that this deceleration is directly proportional to ρ and is given by

$$\Omega = \frac{8\pi\gamma}{3H_0^2}\rho_0 \tag{9–35}$$

From equations (9–15), (9–29), and (9–34), we then find

$$\frac{dl}{dz} = \frac{c}{(1+z)H(z)} \tag{9–36}$$

From equations (9–7) (in the case where $\Lambda = 0$), (9–26), and (9–35), it follows that $H(z)$ is given by

$$H = \left[\frac{8\pi\gamma\rho}{3} - \frac{kc^2}{R^2}\right]^{1/2}$$

$$= \left[H_0^2\Omega(1+z)^3 - \frac{kc^2}{R_0^2}(1+z)^2\right]^{1/2} \tag{9–37}$$

since

$$\rho = \frac{\text{const}}{R^3} = \rho_0(1+z)^3 \tag{9–38}$$

Substituting equation (9–37) into equation (9–36), we then find that

$$\frac{dl}{dz} = \frac{c}{H_0(1+z)^2}\left[(1+z)\Omega - \frac{kc^2}{R_0^2H_0^2}\right]^{-1/2} \tag{9–39}$$

But, from equation (9–37) for $t = t_0$, we obtain the identity

$$-\frac{kc^2}{R_0^2H_0^2} = 1 - \Omega \tag{9–40}$$

so that by substituting equation (9–40) into equation (9–39), we find an expression for dl/dz in the useful form

$$\frac{dl}{dz} = \frac{c}{H_0(1+z)^2(1+\Omega z)^{1/2}} \tag{9–41}$$

Equation (9–41) reduces to equation (9–30) in the case where $\Omega = 1$ and to equation (9–33) when $\Omega \to 0$.

If we specify the value of $\sim 10^{28}$ cm for c/H_0 (which is determined from astronomical observations for $z \lesssim 3 \times 10^{-3}$ to only ± 50 percent), and if we express ρ_0 in terms of an atomic density of n_0 atoms/cm^3, we may use equation (9–35) to define a critical density n_c such that

$$n_c = \frac{3H_0^2}{8\pi\gamma} \simeq 10^{-5}\ \text{cm}^{-3} \tag{9–42}$$

and

$$\Omega = n_0/n_c \tag{9–43}$$

It then follows from equations (9–40) and (9–43) that if the mean density of the universe is greater than n_c, the universe will have a positive curvature; if $n_0 < n_c$, the universe will have a negative curvature, and in the case where $n_0 = n_c$, we have a Euclidean universe of zero curvature (the Einstein-de Sitter model we considered earlier).

We will not consider here models of a still more general nature where $\Lambda \neq 0$, since there is at present no empirical need to do so. However, it is interesting to note in passing that models with $\Lambda \neq 0$ can lead to states of metastable equilibrium in which the universe can maintain a static phase at a fixed value of R for an indefinite period of time. This static model of the universe is known as the Lemaitre model.